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Ponder this!

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Solutions to Problem Set 2

From the work by John Wallis (1655)

1. Prove the Wallis product

$$\frac{\pi}{2} = \frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\ldots\cdot\frac{2n}{2n-1}\cdot\frac{2n}{2n+1}\cdot\ldots$$

or

$$\frac{\pi}{2} = \lim_{n\to\infty}\frac{2\cdot 2\cdot 4\cdot 4\cdot\ldots\cdot 2n\cdot 2n}{1\cdot 3\cdot 3\cdot 5\cdot\ldots\cdot(2n-1)\cdot(2n+1)}\,.$$

Solution The following inequalities hold for $0 < x < \frac{\pi}{2}$

$$\sin^{2n+1} x < \sin^{2n} x < \sin^{2n-1} x.$$

Integrating them in the interval $[0, \dfrac{\pi}{2}]$ we have

$$\int_0^{\frac{\pi}{2}} \sin^{2n+1} x\,dx < \int_0^{\frac{\pi}{2}} \sin^{2n} x\,dx < \int_0^{\frac{\pi}{2}} \sin^{2n-1} x\,dx.$$

Since

$$\int_0^{\frac{\pi}{2}} \sin^m x\,dx = \begin{cases} \frac{(m-1)!!}{m!!}\frac{\pi}{2} & \text{for even}\,m \\ \frac{(m-1)!!}{m!!} & \text{for odd}\,m \end{cases}$$

and taking into account inequalities above we obtain

$$\frac{2n!!}{(2n+1)!!} < \frac{(2n-1)!!}{2n!!}\frac{\pi}{2} < \frac{(2n-2)!!}{(2n-1)!!}$$

or

$$\left[\frac{2n!!}{(2n-1)!!}\right]^2 \frac{1}{2n+1} < \frac{\pi}{2} < \left[\frac{2n!!}{(2n-1)!!}\right]^2 \frac{1}{2n}.$$

Since the difference between RHS and LHS

$$\frac{1}{2n(2n+1)}\left[\frac{2n!!}{(2n-1)!!}\right]^2 < \frac{1}{2n}\frac{\pi}{2}$$

and approaches 0 when $n \to \infty$, $\frac{\pi}{2}$ is a common value for both limits, RHS and LHS.

Thus,

$$\frac{\pi}{2} = \lim_{n\to\infty}\left[\frac{2n!!}{(2n-1)!!}\right]^2 \frac{1}{2n+1}$$

or

$$\frac{\pi}{2} = \lim_{n\to\infty} \frac{2\cdot 2\cdot 4\cdot 4\cdot \ldots \cdot 2n\cdot 2n}{1\cdot 3\cdot 3\cdot 5\cdot \ldots \cdot (2n-1)\cdot (2n+1)}.$$

Proposed by Michel Bataille

2. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$xyf(x+y) = y(x+3y)f(x) + x(y+3x)f(y)$$

for all real numbers x, y.

Solution It is easily checked that every function $x \mapsto cx^3$ (where c is a real constant) is a solution. We show that there are no other solutions.
Let f be any solution. Taking $x = 0, y = 1$ in the functional equation gives $f(0) = 0$. Then, with $y = -x$, we obtain

$$0 = 4x^2(f(x) + f(-x))$$

so that $f(-x) = -f(x)$ for nonzero x. It follows that f is odd.
Now, substituting $x + y$ for x and $-y$ for y yields

$$-y(x+y)f(x) = -y(x-2y)f(x+y) - (x+y)(3x+2y)f(y)$$

so that

$$(2y-x)(xyf(x+y)) = x(x+y)(3x+2y)f(y) - xy(x+y)f(x)$$

for all real numbers x, y. Using the functional equation to eliminate $xyf(x+y)$ leads to

$$yf(x)[(x+3y)(2y-x)+x(x+y)] = xf(y)[(x+y)(3x+2y)+(x-2y)(y+3x)]$$

or, after an easy calculation: $\quad 6y^3f(x) = 6x^3f(y) \quad$ and so

$$f(x) = x^3f(1)$$

for all real x. This completes the proof. (Michel Bataille)

Particular case $n = 4$ of the Shapiro inequality

3. If a, b, c, $d > 0$ prove that

$$\frac{a}{b+c} + \frac{b}{c+d} + \frac{c}{d+a} + \frac{d}{a+b} \geq 2\,.$$

Solution Since $4xy = (x+y)^2 - (x-y)^2$ it follows that $\dfrac{(x+y)^2}{4} \geq xy$ for $x, y \geq 0$. Now

$$\begin{aligned}\frac{a}{b+c} + \frac{b}{c+d} + \frac{c}{d+a} + \frac{d}{a+b} &= \frac{a(d+a)+c(b+c)}{(b+c)(d+a)} + \frac{b(a+b)+d(c+d)}{(c+d)(a+b)}\\ &\geq \frac{4(a^2+b^2+c^2+d^2+ad+bc+ab+cd)}{(a+b+c+d)^2}\\ &= \frac{2\left[(a+b+c+d)^2+(a-c)^2+(b-d)^2\right]}{(a+b+c+d)^2}\\ &\geq 2\,.\end{aligned}$$

4. Let for every $n \in \mathbb{N}$ $\;a_n$ and b_n be integers from

$$\left(1+\sqrt{3}\right)^n = a_n + b_n\sqrt{3}\,.$$

Evaluate

$$\lim_{n\to\infty} \frac{a_n}{b_n}\,.$$

Solution Since $\left(1-\sqrt{3}\right)^n = a_n - b_n\sqrt{3}$, for $n \geq 1$ we have

$$\begin{aligned}a_n &= \frac{1}{2}\left(\left(1+\sqrt{3}\right)^n + \left(1-\sqrt{3}\right)^n\right),\\ b_n &= \frac{1}{2\sqrt{3}}\left(\left(1+\sqrt{3}\right)^n - \left(1-\sqrt{3}\right)^n\right),\end{aligned}$$

which yields $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \sqrt{3}$.

5. Prove that $x = 2$ is a unique solution on the set $\mathbb{R}$ to the equation

$$3^x + 4^x = 5^x \,.$$

Solution Let $f(x) = \left(\frac{3}{5}\right)^x + \left(\frac{4}{5}\right)^x - 1, \qquad x \in \mathbb{R}$. Then, the initial equation is equivalent to $f(x) = 0$. Assume the latter has two roots. This implies that by Rolle's theorem there exists such a point Θ that $f'(\Theta) = 0$.

However,

$$f'(x) = \left(\frac{3}{5}\right)^x \ln\frac{3}{5} + \left(\frac{4}{5}\right)^x \ln\frac{4}{5} < 0, \qquad x \in \mathbb{R}$$

leads to a contradiction. Thus, $x = 2$ is the only solution and we are done.