Ponder this

Problem set 1 solutions ASMJ 22 (1) 2008

Edited by Oleksiy Yevdokimov
University of Southern Queensland
yevdokim@usq.edu.au

1. Prove that for each positive integer $n \geq 3$, a number 2^n can be represented as $2^n = 7x^2 + y^2$ where x and y are both odd numbers. (in memoriam of Leonard Euler)

Solution

We prove this statement by induction. For $n = 3$ it is true. Assume that the property is true for a certain *n*, i.e. $2^n = 7x^2 + y^2$, where x and y are both odd numbers. Then, for pairs

$$\left\{A = \frac{1}{2}(x-y),\ B = \frac{1}{2}(7x+y)\right\} \text{ and } \left\{C = \frac{1}{2}(x+y),\ D = \frac{1}{2}(7x-y)\right\} \text{ we have } 2^{n+1} = 7A^2 + B^2$$

and $2^{n+1} = 7C^2 + D^2$, respectively. A and B are either odd or even simultaneously. Indeed, if $A = \frac{1}{2}(x-y) = l$ is odd, then $B = \frac{1}{2}(7x+y) = \frac{1}{2}(7y + 14l + y) = 4y + 7l$ must be odd. If A is even, then B is even, respectively. The same property is valid for C and D. Moreover, if $A = \frac{1}{2}(x-y)$ is odd, then $C = \frac{1}{2}(x+y)$ is even, and vice versa. This means that both numbers in one of the pairs are odd.

2. There are two non-congruent triangles, T_1 and T_2. For any pair of sides of T_1, there is a pair of sides of T_2 with the same sum of their lengths. Likewise, for any pair of sides of T_2, there is a pair of sides of T_1 with the same sum of their lengths. The triangle T_1 has two sides of lengths 25 and 35, respectively. Prove that the triangles have different perimeters. Find the triangle with the smaller perimeter and evaluate that perimeter. (G.Galperin)

Solution

Let 25, 35, α and x, y, z be lengths of sides of triangles T_1 and T_2 respectively. Suppose without loss of generality that $x + y = 60$. We prove that the triangles have different perimeters by contradiction. Assume the opposite, i.e. both triangles have the same perimeter: $25 + 35 + \alpha = x + y + z$. This yields, $\alpha = z$. The sum $35 + z$ could be equal either $z + y$, $z + x$ or 60. The first two cases lead to a contradiction immediately since the triangles are non-congruent. The third case gives $z = 25$, which guarantees that both triangles are congruent with sides 25, 35, 25. Again, we get a contradiction. Hence, the triangles have different perimeters.

We note that either $25 + \alpha = 60$ or $35 + \alpha = 60$. Indeed, $25 + \alpha = x + z$ and $35 + \alpha = y + z$ yield $x + 10 = y$ with $x = 25$ which is impossible since $\alpha \neq z$. The other case $25 + \alpha = y + z$ can be treated in similar way. Thus, for $\alpha = 35$ we have perimeter of T_1 $25 + 35 + 35 = 95$. Using conditions $x + z = 60$ or $x + z = 70$ and $y + z = 60$ or $y + z = 70$ we can compose four systems of simultaneous equations, which give the only possible result for T_2: 30, 30, 40 and perimeter 100. For $\alpha = 25$ we have perimeter of T_1 $25 + 35 + 25 = 85$. Again, using conditions $x + z = 50$ or $x + z = 60$ and $y + z = 50$ or $y + z = 60$ we can compose four systems of simultaneous equations, which give the only possible result for T_2: 30, 30, 20 and perimeter 80. Therefore, the triangle T_2 has the smallest perimeter 80.

3. A square grid on the Euclidean plane consists of all points (m,n), where m and n are integers. Is it possible to cover all grid points by an infinite family of discs with non-overlapping interiors:

(a) if each disc in the family has a radius of 1.3?

(b) if each disc in the family has a radius of at least 5? (G.Galperin)

Solution

(a) It is possible. Each pentagon on Figure 1 can be covered by a disc with radius 1.3 and no overlapping occurs.

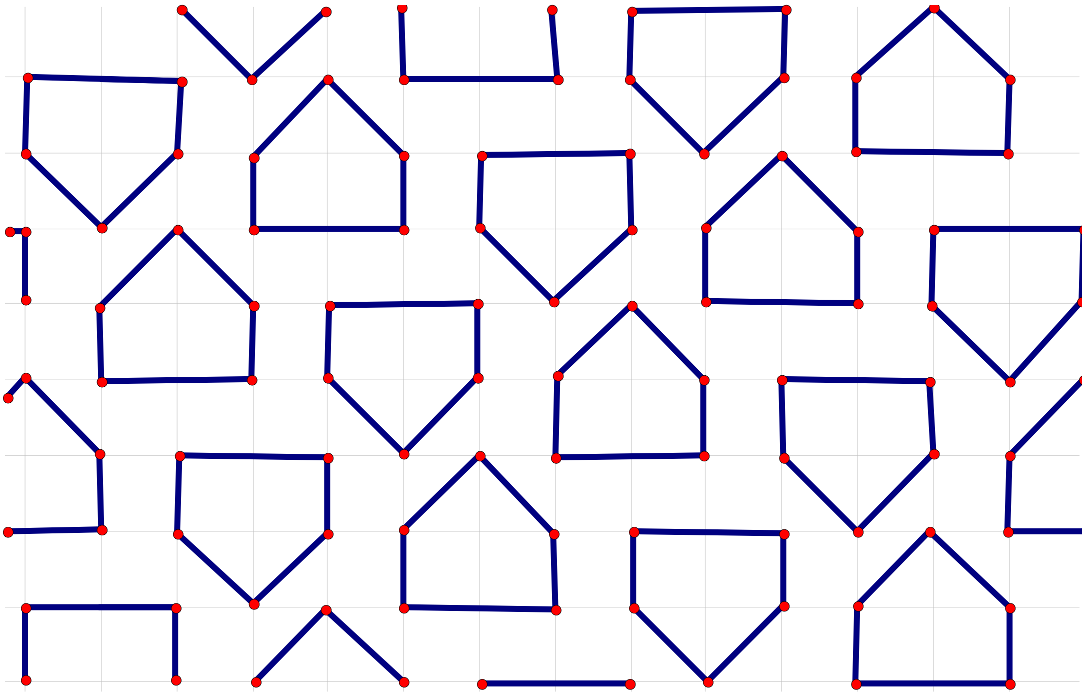

Figure 1.

(b) It is not possible. The proof is by contradiction. Suppose that such a covering family F exists. Let $D(P,\rho)$ denote the disc with centre P and radius ρ. Start with an arbitrary disc $D(O,r)$ that does not overlap any member of F. Then $D(O,r)$ covers no grid point. Take $D(O,r)$ to be maximal in the sense that any further enlargement would cause it to violate the non-overlap condition. Then $D(O,r)$ is tangent to at least three discs in F. Observe that there must be two of these tangent discs, say $D(A,a)$ and $D(B,b)$, such that $\angle AOB \le 120°$. By the law of cosines applied to the triangle ABO,

$$(a+b)^2 \le (a+r)^2 + (b+r)^2 + (a+r)(b+r),$$

which yields,

$$ab \le 3(a+b)r + 3r^2, \text{ and thus } 12r^2 \ge (a-3r)(b-3r).$$

Note that $r < 1/\sqrt{2}$ because $D(O,r)$ covers no grid point, and $(a-3r)(b-3r) \ge (5-3r)^2$ because each disc in F has radius at least 5. Consequently, $2\sqrt{3}r \ge (5-3r)$ which gives

$5 \le (3+2\sqrt{3})r < (3+2\sqrt{3})/\sqrt{2}$ and thus $5\sqrt{2} < 3+2\sqrt{3}$. Squaring this inequality yields $50 < 21+12\sqrt{3}$, hence $29 < 12\sqrt{3} < 12 \cdot 2 = 24$. This contradiction completes the proof.

Remark. The above argument shows that no covering family exists where each disc has radius greater than $(3+2\sqrt{3})/\sqrt{2}$. In the other direction, there exists a covering family in which each disc has radius $\sqrt{13}/2 \approx 1.802$. Take discs with this radius centered at grid points of the form $(2m+4n, 3m)$, where m and n are integers. Then any grid point is within $\sqrt{10}/2$ of one of the centers and the distance between any two centres is at least $\sqrt{13}$. The extremal radius of a covering family is unknown. (G.Galperin)

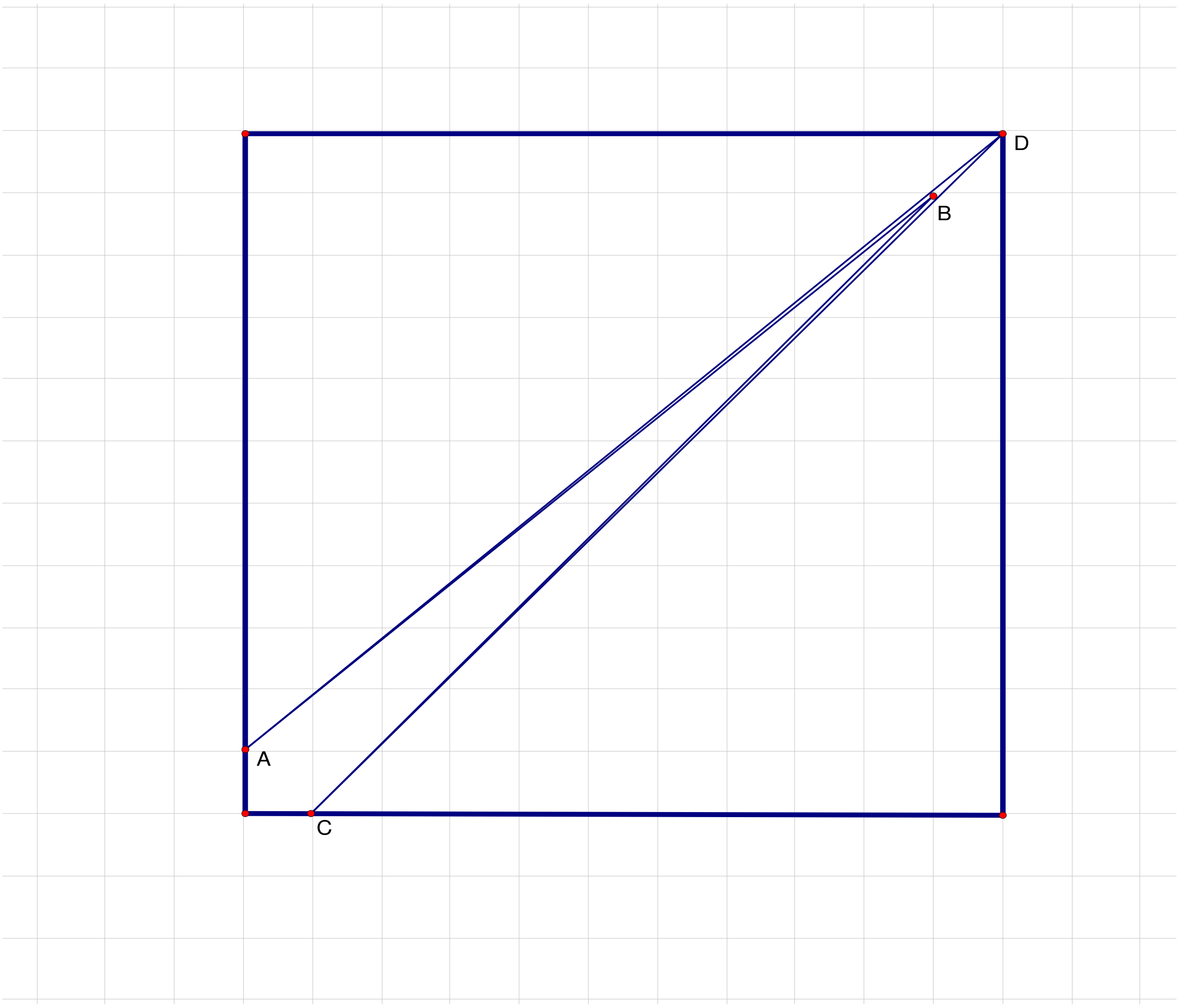

Figure 2.

4. Janet and Jack play a game on an $n \times n$ grid. Each player in turn draws a polygon (not necessary a convex one) with vertices at the grid points (i.e., points (i, j), where i, j are integers) and area 1.

Janet goes first and then the players alternate. Each new polygon cannot share any common point with polygons drawn before. Janet wins if Jack cannot draw a polygon and vice versa. Find, with proof, a winning strategy for one of the players. (B.Rublev)

Solution

Here is the winning strategy for Janet. If she starts drawing a polygon on Figure 2, she will be the winner every time. Indeed, the polygon $ABCD$ has area 1 (since $1 = S_{ACD} - S_{ABC}$). All other polygons can be drawn on one or other side of the grid since each new polygon cannot share any common point with polygons drawn before, in particular with $ABCD$. Therefore, using symmetric strategy Janet will win.

5. Let points B and C of a semicircle with diameter DE satisfy $BD + CE = DE$. If lines BD and CE intersect at A, prove that

$$\frac{2}{BC} \le \frac{1}{AC} + \frac{1}{AB}. \text{ (M.Bataille)}$$

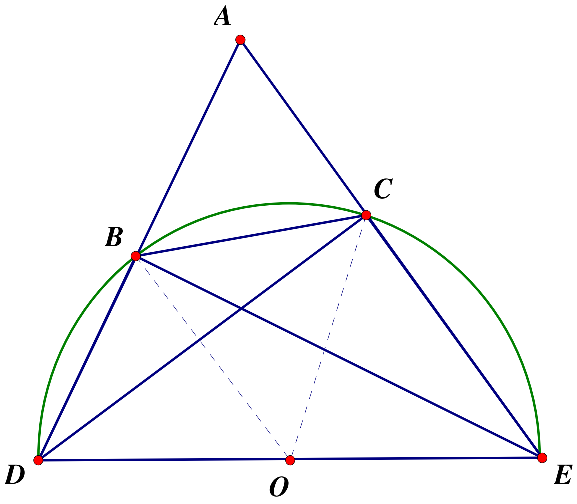

Figure 3.

Solution

First, note that $\angle DBE = \angle DCE = 90°$ (Figure 3). Let $B = \angle ABC$, $C = \angle ACB$ and let O be the midpoint of DE. Observing that B, D, C, E are on the same semicircle, we have $C = 180° - \angle BCE = \angle BDE$ and similarly $B = \angle CED$.

Since $\cos\angle BDE = \dfrac{BD}{DE}$ and $\cos\angle CED = \dfrac{CE}{DE}$, the hypothesis rewrites as

$$\cos B + \cos C = 1 \qquad (*).$$

Now, setting as usual $BC = a$, $CA = b$, $AB = c$, the law of cosines gives

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}, \quad \cos B = \frac{a^2 + c^2 - b^2}{2ac}$$

And from (*) we obtain

$$2abc = c\left(a^2 + b^2 - c^2\right) + b\left(a^2 + c^2 - b^2\right).$$

It follows that

$$2abc = a^2(b+c) - (b-c)^2(b+c) \leq a^2(b+c)$$

that is,

$$\frac{2}{a} \leq \frac{1}{b} + \frac{1}{c}$$

as required.

It is readily seen that equality holds if and only if $BD = CE = \frac{DE}{2}$. (M.Bataille)